Greek Letters

α = external void fraction
β = intraparticle void fraction
γ = function defined by Equation (16)
$\delta_o, \delta_1, \delta_2, \delta_e, \delta_i$ = groups of constants defined by Equations (26), (27), (28), (36), and (37)
λ = expression defined by Equation (18)
μ'_n, μ_n = nth absolute and central moment, respectively, Equations (19), (21)
τ = reduced time defined by Equation (41)
θ_{propane} = coverage of the adsorbent surface, fraction of a monolayer

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Manuscript received July 6, 1967; revision received November 9, 1967; paper accepted November 30, 1967.

Two Phase Friction Factor for Para-Hydrogen between One Atmosphere and the Critical Pressure

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The Martinelli model for pressure drop in flowing two-phase systems has been examined in detail for para-hydrogen from 1 atm. to its critical pressure. A method for obtaining the Martinelli Φ term, two phase friction factor, at intermediate pressures is presented. Φ^2 can be expressed as a function of pressure and quality. The Martinelli β term, ratio of actual vapor or gas area to an idealized cylindrical area, is found to vary greatly and to indicate first an increase in gas-liquid interface to gas volume ratio and then a decrease with increasing pressure. Some experimental data are examined in light of this development.

A need exists for a semi-empirical, theoretical method to predict pressure drop for flowing two phase systems at pressures other than those for which data is readily obtained. Most often for one component systems, data are limited to near atmospheric conditions or at best to a few pressures intermediate between one atmosphere and the critical condition. This development offers a means of determining the two phase pressure drop for one component systems over the entire regime of gas-liquid coexistence, provided a good estimate can be made of either α or β as defined for the Martinelli, et al. model (*1*). This development has been performed for para-hydrogen.

Φ VS. X AT INTERMEDIATE PRESSURES

Figure 1 gives the Φ, R_l, X relationships as presented by Lockhart and Martinelli (*2*). Only Φ for the liquid, turbulent-turbulent system will be considered; hence no subscripts will be carried on Φ or other terms if not needed.

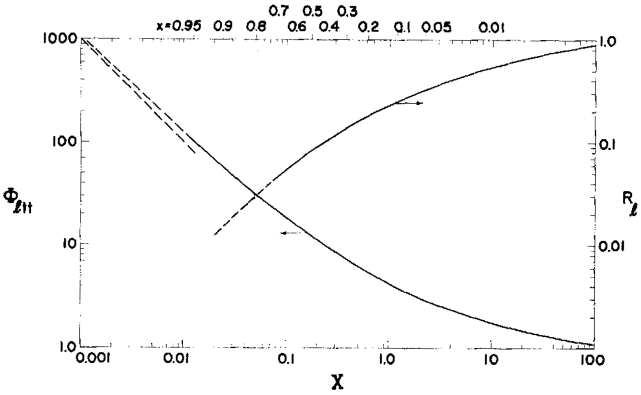

Fig. 1. Martinelli Φ and R_l vs. X correlation.

Figure 2 gives χ vs. quality or weight fraction gas with pressure as the parameter ranging from one to 12.759 atm., the critical pressure of para-hydrogen. Densities of hydrogen were obtained from the work of Roder, et al. (*3*) and computed by the Farmer method (*4*). The viscosities were obtained from Rogers, et al. (*5*). The definition of χ is given (*1*) as

$$\chi = X^{1.11} = \left(\frac{\rho_g}{\rho_l}\right)^{0.555} \left(\frac{\mu_l}{\mu_g}\right)^{0.111} \left(\frac{1-x}{x}\right) \qquad (1)$$

The exponents take into account, among other things, the single phase friction factor which is inversely proportional to the Reynolds number to the 0.2 power.

Figure 3 is a plot of α and β vs. χ for 1 atm. conditions. α and β are defined respectively (*1*) as the ratio of liquid and vapor area perpendicular to the direction of flow to what they would be if they were ideally cylindrical in cross section. See Equations (13) and (15). They are computed from the equations given elsewhere (*2*).

$$\alpha = R_l^4 \ \Phi^{3.33} \qquad (2)$$

$$\beta = R_g^4 \ (\Phi X)^{3.33} \qquad (3)$$

Cartesian coordinate graphs of α and β vs. χ were made and the slopes determined therefrom by graphical means. These slopes versus quality are presented in Figures 4 and 5.

The change in χ with pressure at a number of qualities was computed by a straight line difference method from χ at two slightly different pressures near 1 atm. This provided $\partial\chi/\partial P$, which when multiplied by the slopes of Figures 4 and 5, gave $\partial\alpha/\partial P$ and $\partial\beta/\partial P$ at 1 atm. for a number of values of quality.

The equation for Φ is

$$\Phi = [\alpha^{0.25} + \beta^{0.25}/\chi^{0.75}]^{1.2} \qquad (4)$$

and at the critical condition has been shown by Martinelli and Nelson (*6*) to be

$$\Phi = \left(\frac{1}{1-x}\right)^{0.9} = \left(\frac{\chi+1}{\chi}\right)^{0.9} \qquad (5)$$

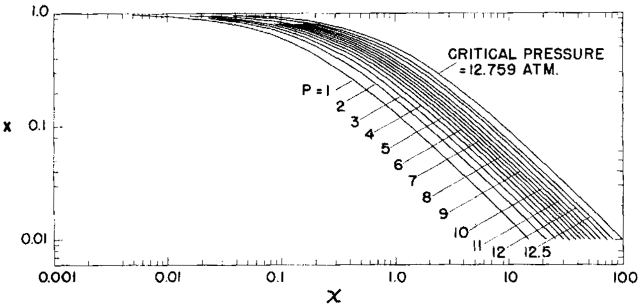

Fig. 2. Quality vs. χ with pressure as parameter.

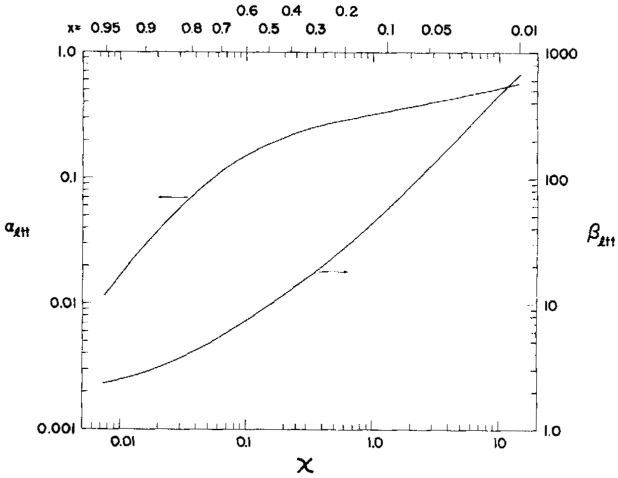

Fig. 3. α and β vs. χ at 1 atm.

The derivative of Equation (4) with respect to pressure at constant quality gives

$$\left(\frac{\partial\Phi}{\partial P}\right)_x = 1.2 \qquad {}^{25}$$

$$+ \beta^{0.25}\left(\frac{\rho_l}{\rho_g}\right)^{0.416}\left(\frac{\mu_g}{\mu_l}\right)^{0.083}\left(\frac{x}{1-x}\right)^{0.75}\Bigg]^{0.2}$$

$$\Bigg\{0.25\ \alpha^{-0.75}\left(\frac{\partial\alpha}{\partial P}\right)_x + 0.25\ \beta^{-0.75}\ \chi^{-0.75}\left(\frac{\partial\beta}{\partial P}\right)_x$$

$$+\beta^{0.25}\left(\frac{x}{1-x}\right)^{0.75}$$

$$\Bigg[0.416\ \rho_l^{-0.584}\ \rho_g^{-0.416}\left(\frac{\mu_g}{\mu_l}\right)^{0.083}\left(\frac{\partial\rho_l}{\partial P}\right)_x$$

$$-\ 0.416\ \rho_l^{0.416}\ \rho_g^{-1.416}\left(\frac{\mu_g}{\mu_l}\right)^{0.083}\left(\frac{\partial\rho_g}{\partial P}\right)_x$$

$$+\ 0.083\left(\frac{\rho_l}{\rho_g}\right)^{0.416}\mu_g^{-0.917}\ \mu_l^{-0.083}\left(\frac{\partial\mu_g}{\partial P}\right)_x$$

$$-\ 0.083\left(\frac{\rho_l}{\rho_g}\right)^{0.416}\mu_g^{0.083}\ \mu_l^{-1.083}\left(\frac{\partial\mu_l}{\partial P}\right)_x\Bigg]\Bigg\} \qquad (6)$$

Equation (5) can be considered to be either the limit of Φ_{ltt} at the critical point as shown previously (*6*) or,

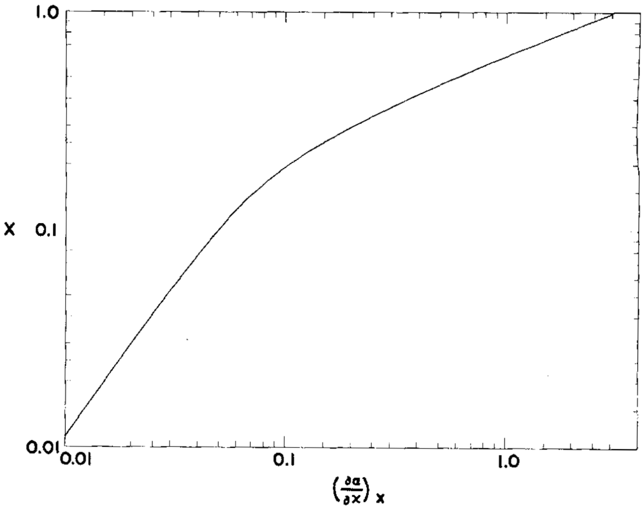

Fig. 4. Quality vs. $(\partial\alpha/\partial\chi)_x$ at 1 atm.

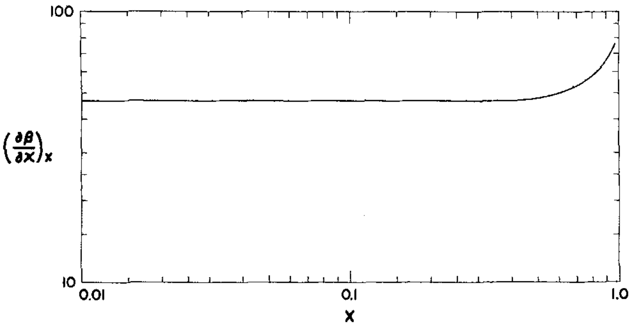

Fig. 5. $(\partial\beta/\partial x)_x$ vs. quality at 1 atm.

more generally, the value of Φ_{ltt} for expressing pressure drop in the entire liquid single phase region. The functional form of Φ_{ltt} and its pressure derivative can be obtained as the solution to a boundary value problem along the vapor-liquid coexistence curve which includes all pressures up to and including the critical point. In the manner characterized by Martinelli and Nelson (*6*), the continuity condition at the boundary for pressure drop gives

$$\left(\frac{\Delta P}{\Delta L}\right)_{TP} = \left(\frac{\Delta P}{\Delta L}\right)_l \Phi^2_{ltt} = \left(\frac{\Delta P}{\Delta L}\right)_{SP} (1-x)^{1.8}\,\Phi^2_{ltt} \qquad (6a)$$

where $(1-x)^{1.8}\,\Phi^2_{ltt} = 1$ along the coexistence curve for which $x = 0$ for all liquid and $0 < x < 1$ at the critical point. The mathematical concept of Φ in the entire single phase region is preserved as an expression in terms of quality. Similarly, Φ_{gtt} for the gaseous single phase region can be shown to be

$$\Phi_{gtt} = x^{-0.9} \qquad (6b)$$

Since Equation (5) is not limited to a specific pressure (that is, the critical point) with this interpretation, its pressure derivative can be obtained and then the substitutions, $\rho_g = \rho_l$ and $\mu_g = \mu_l$ made at the critical point to give

$$\left(\frac{\partial\Phi}{\partial P}\right)_x = \frac{x}{(1-x)^{0.9}}\left\{\frac{1}{2\rho_g}\left[\left(\frac{\partial\rho_l}{\partial P}\right)_x - \left(\frac{\partial\rho_g}{\partial P}\right)_x\right] + \frac{1}{10\,\mu_g}\left[\left(\frac{\partial\mu_g}{\partial P}\right)_x - \left(\frac{\partial\mu_l}{\partial P}\right)_x\right]\right\} \qquad (7)$$

Comparison of Equation (6) with (7) indicates that $\partial\alpha/\partial P$ and $\partial\beta/\partial P$ must go to zero at the critical pressure. With these simplifications Equations (4), (5), (6), and (7) can be solved simultaneously by first equating (4) to (5) and (6) to (7) to satisfy the boundary continuity conditions. The final solution yields the results at the critical condition that

$$\alpha = 1 - x \qquad (8)$$

$$\beta = x \qquad (9)$$

Since Φ depends only upon quality, inspection of Equation (5) also indicates that along the coexistence curve and at the critical condition, the change in Φ with pressure at constant quality is zero. Equation (7) gives the same result in the limit of the property derivatives. Equation (6) is used to calculate the change in Φ with pressure at 1 atm. The same results are obtainable from two alternate considerations of the continuity conditions. The values of α and β as well as their pressure derivatives can be determined as follows. From Martinelli, et al. (*1*), equations relating Φ with α and β can be deduced. These are

$$\Phi^2_{ltt} = \frac{1}{\alpha^{1.8}}\left(\frac{D_p}{D_l}\right)^{4.8} \qquad (8a)$$

$$\Phi^2_{gtt} = \frac{1}{\beta^{1.8}}\left(\frac{D_p}{D_g}\right)^{4.8} \qquad (9a)$$

The diameter ratios are unity for single phase conditions including the critical point. Equations (6*a*) and (6*b*) with (8*a*) and (9*a*) then identify α and β to be the same as Equations (8) and (9). Equations (5), (8), and (9) must hold at all pressures under single phase conditions including the critical point and are independent of pressure. Therefore $(\partial\Phi/\partial P)_x$, $(\partial\alpha/\partial P)_x$, and $(\partial\beta/\partial P)_x$ are all zero at the boundary between single phase and two phase regions.

Equations (8) and (9) also could have been deduced by the requirement that for single phase conditions the volume fractions of the phases must equal their weight fractions, thus

$$R_l = 1 - x \qquad (8b)$$

$$R_g = x \qquad (9b)$$

This result is identical with that obtained elsewhere (*6*) from consideration of the pressure drop associated with acceleration at the critical point. Equations (8*b*), (9*b*), (2), (3), and (5) then combine to give (8) and (9).

The limiting or boundary value form of the various Martinelli model parameters and their pressure derivatives have been obtained from several approaches. In the first instance the simultaneous solution of Equations (4) through (7) requires that Equations (5) and (7) be written in terms of viscosities and densities at the critical point. This gives rise to two identically formed indeterminate expressions in terms of differences of the density and viscosity derivatives. In the limit of the pressure derivative differences of the densities and viscosities at the critical point, the coefficients of these expressions are equated after having set $\partial\alpha/\partial P$ and $\partial\beta/\partial P$ equal to zero and together with Equations (4) and (5) are solved for α and β. The alternate methods depend upon insight to the physical significance of D_l, D_g, R_l, and R_g along the coexistence curve. It should be clear that the Martinelli model yields the limiting values of the parameters from several approaches and that a somewhat broader interpretation than usual is applied herein.

The above information, (Equations (2) and (8), Figure 4, and $\partial\alpha/\partial P = 0$ at the critical point) permits the formulation of an α map. This is shown in Figure 6. Lockhart and Martinelli (*2*) comment on the fact that α does not exceed one for the systems they studied. This limited value would indicate that the liquid phase characteristics might be well behaved for a number of systems and that an α map is easier to estimate than is a β map. For parahydrogen, the computed values and pressure derivatives of α are shown at 1 atm. and at the critical pressure. These are of such magnitude that α for hydrogen most likely does not exceed one for any pressure or quality. The intermediate α values are interpolated graphically with curves drawn through the values tangent to the slopes. It was found that subsequent calculations are not overly sensitive to minor changes in either the value or slope of α; hence, inaccuracies are not serious, provided a well behaved situation as depicted in Figure 6 does exist. Figure 6, the α mapping, becomes the key to the entire subsequent development. This mapping, together with the saturated equilibrium assumption, contains the only considered assumption other than those basic to the model being used.

Equations (4) through (7) provide the means for computing the values of Φ and $\partial\Phi/\partial P$. Values for Φ at 1 atm. were obtained from Figure 1. Figure 7 shows the result of a graphical interpolation between the Φ values at 1 atm. and the critical pressure for $x = 0.9$ with the curve tangent to the slopes at the end values.

Equation (4) can be solved for β to give

$$\beta = (\Phi^{0.833} - \alpha^{0.25})^4 \chi^3 \qquad (10)$$

The derivative of Equation (10) at constant x is

$$\left(\frac{\partial \beta}{\partial P}\right)_x = 4B^3\chi^3 \left(\frac{\partial \beta}{\partial P}\right)_x + 3B^4\chi^2 \left(\frac{\partial \chi}{\partial P}\right)_x \qquad (11)$$

in which $B = \Phi^{0.833} - \alpha^{0.25}$. Equations (6), (10), and (11) together with Figures 2 and 6 form a complete set for obtaining the desired Φ vs. P curve by iterative solution.

For graphical solution, the values of Φ and $\partial\Phi/\partial P$ at constant x are obtained from Figure 1 and are computed from Equations (5) and (6) for 1 atm. and the critical pressure. The Φ vs. P curve is then drawn and values of β are computed from Equation (10) for several values of pressure. Curves as shown in Figures 8, 9, and 10 are thus determined. The values of $\partial\beta/\partial P$ at 1 atm., as obtained graphically, and at the critical pressure from cross comparison of Equations (6) and (7) were useful for constructing the β, pressure curves. Slopes from the β vs. pressure plots were then obtained graphically. These were used together with the computed β values and the α map values in Equation (6) to compute $\partial\Phi/\partial P$. These computed slopes were compared with those measured graphically from the $\Phi - P$ curve. If the $\Phi - P$ curve slope comparison was unfavorable, adjustments to the curve were made and the solution process repeated until agreement to within a few percent was obtained. Pressure derivatives of density and viscosity for Equation (6) were obtained by the straight line difference method from values of the properties at two slightly different pressures near the pressure of interest. Typical data for final $\Phi - P$ curve slope comparisons are shown in Table 1 for $x = 0.3$ and $x = 0.9$.

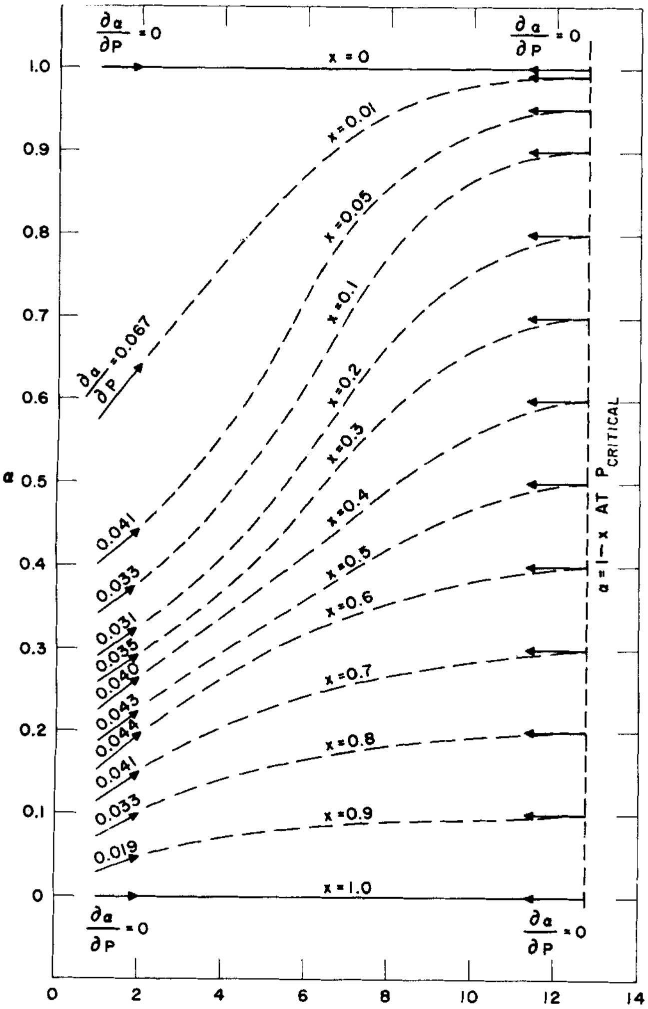

Fig. 6. α vs. pressure with quality as parameter.

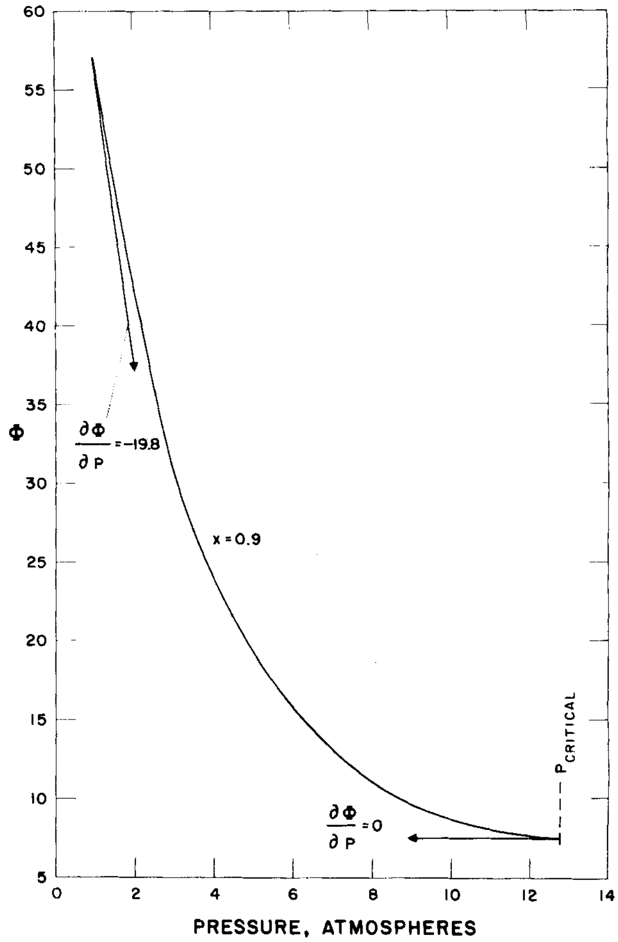

Fig. 7. Φ vs. pressure, $x = 0.9$.

TABLE 1. $\partial\Phi/\partial P$ AS COMPUTED AND MEASURED FOR ITERATIVE CALCULATION

	$x = 0.3$		$x = 0.9$	
Pressure	Calc.	Graphical	Calc.	Graphical
1.5 atm.	−1.60	−1.60	−14.03	−14.4
3 atm.	−0.98	−0.96	−8.49	−8.8
5 atm.	−0.51	−0.53	−4.26	−4.1
7 atm.	−0.33	−0.32	−2.48	−2.4

The results of the entire procedure are shown in Figures 11 and 12. The latter was constructed from Figure 1 and from a cross plotting of data used to generate Figures 2 and 11. If a temperature or heat flow vs. length profile is known together with the quality at one position for a flowing hydrogen system and the saturated phase equilibrium assumption is imposed, then Figure 11 can be used directly to obtain the pressure drop.

Data for curves like those of Figure 11 were computed from qualities of 0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 0.8, and 0.9. From the $\Phi - P$ curves, values of Φ were tabulated for pressures of 1 through 11 atm., and the critical pressure (Table 2).

This table constitutes the complete solution for the saturated equilibrium two-phase friction factor for hydrogen in accord with the Martinelli model. This value is defined elsewhere (*1*) as

$$\Phi^2 = \left(\frac{\Delta P}{\Delta L}\right)_{TP} \Bigg/ \left(\frac{\Delta P}{\Delta L}\right)_l \tag{12}$$

where $(\Delta P/\Delta L)_{TP}$ is the two phase pressure drop and $(\Delta P/\Delta L)_l$ is the pressure drop for the liquid phase only flowing. The Φ^2 term is thus based upon the liquid phase and as stated earlier is only for a completely turbulent system.

R_l values for the $R_l - X - P - x$ surface were computed with Equation (2) using the α values of Figure 6 and Φ values from Table 2.

Table 2. Φ as a Function of P and x

Pressure in Atmospheres

x	1	2	3	4	5	6	7	8	9	10	11	P_{crit}
0.01	1.70	1.55	1.40	1.28	1.20	1.14	1.09	1.06	1.04	1.02	1.01	1.008
0.05	2.77	2.34	2.02	1.77	1.58	1.42	1.31	1.20	1.12	1.09	1.05	1.045
0.1	3.70	3.11	2.64	2.27	1.96	1.71	1.50	1.36	1.25	1.17	1.12	1.098
0.3	7.30	5.70	4.55	3.72	3.07	2.60	2.25	1.96	1.73	1.53	1.44	1.358
0.5	12.2	9.15	7.00	5.50	4.44	3.65	3.03	2.59	2.27	2.05	1.95	1.864
0.7	21.2	16.3	12.0	9.20	7.30	5.95	5.00	4.28	3.75	3.30	3.00	2.955
0.8	31.0	23.1	17.3	13.5	10.7	8.80	7.20	6.00	5.15	4.70	4.40	4.26
0.9	57.0	42.5	30.7	23.9	19.3	15.8	13.1	11.0	9.50	8.60	8.10	7.92

An empirical equation which represents the data of Table 2 and has the correct single phase and critical point limiting values for Φ^2 is

$$\Phi^2 = \left(\frac{1}{1-x}\right)^{1.8} \{1 + x^{0.8187}\,[0.1324(12.759-P) + 0.03966\,(12.759-P)^3]/P^E\} \tag{12a}$$

where $E = 1.896\,x - 2.646\,x^2 + 1.695\,x^3$. Of the 88 tabulated values below the critical pressure, this equation gives 68 values of Φ^2 within 5%, 73 within 10%, 75 within 11.5%, and 88 within 20%.

DISCUSSION

Several aspects of the above development bear examination. These are the 1 atm. values of α and β as computed from Equations (2) and (3), the assumed α map, the resulting β map, the relative significance of various terms contributing to the $\Phi - P$ curves at the higher pressures, the treatment of the $\Phi - P$ curves at the higher pressures, the significance of α and β, the nature of the $\Phi - X - P - x$ surface, and the effect of uncertainties in the physical properties of hydrogen upon the $\Phi - X - P - x$ surface.

The values of α and β at 1 atm. as they are computed from Equations (2) and (3) are particularly sensitive to the volume fractions liquid and gas, R_l and R_g. Wicks, et al. (8) have indicated that holdup data is somewhat better represented by the correlation of Hughmark (9). This is apparent for R_l near 0.2 which corresponds to quality values greater than 0.2 with the disagreement with the Lockhart and Martinelli correlation increasing with decreasing R_l and increasing x.

The term in α, $\alpha^{-0.75}/4\ \partial\alpha/\partial P$, of Equation (6) is of the order of 10% (as a maximum but usually much less) of the contribution to $\partial\Phi/\partial P$ for R_l when R_l is less than 0.2 and x greater than 0.2. Thus the arithmetic mean deviations of Wicks, et al. (8) indicate corrections for small R_l which would reduce to even smaller changes in $\partial\Phi/\partial P$ from Equation (6). In addition, any corrections to α would cause a change in $\partial\alpha/\partial P$ such that some compensation may occur in the term of concern. The first term of Equation (6) is just $\Phi^{0.1667}$ and is never computed from α and β but is obtained graphically. The term in β, $0.25\ \beta^{-0.75}\ \chi^{-0.75}\ \partial\beta/\partial P$, of Equation (6) is of the order of 50% at $x = 0.01$ of the contribution to $\partial\Phi/\partial P$ at 1 atm.; 50% is a maximum and the percentage is smaller at the high values of quality down to about 15% at $x = 0.9$. Thus for the values of R_l and R_g at high quality which are least well predicted, this term in β is much less significant. The other terms of Equation (6) have $\beta^{0.25}$ as a coefficient. The small exponent on β reduces the sensitivity to uncertainties in R_l of these terms contributing to $\partial\Phi/\partial P$.

The correlation of Hughmark for holdup is geometry dependent through the Reynolds and Froude numbers and is not easily adapted to a general correlation of this nature which is independent of pipe diameter.

The computation of β, at intermediate pressures, with Equation (10) involves $\alpha^{0.25}$ as compared with $\Phi^{0.833}$. $\alpha^{0.25}$ is approximately 15% of $\Phi^{0.833}$ for R_l at about 0.2 and $P = 1$; at $P = 9$, $\alpha^{0.25}$ is about 70% of $\Phi^{0.833}$. In Equation (10), uncertainties in α are more important to the calculation of β than they are to $\partial\Phi/\partial P$ in Equation (6). However, as R_l becomes even smaller and harder to predict, the term $\alpha^{0.25}$ becomes much less important to the calculation of β. Also, as the pressure is increased, α approaches its critical pressure value, which is completely determined, and thus the value of α should be better known for the higher pressures.

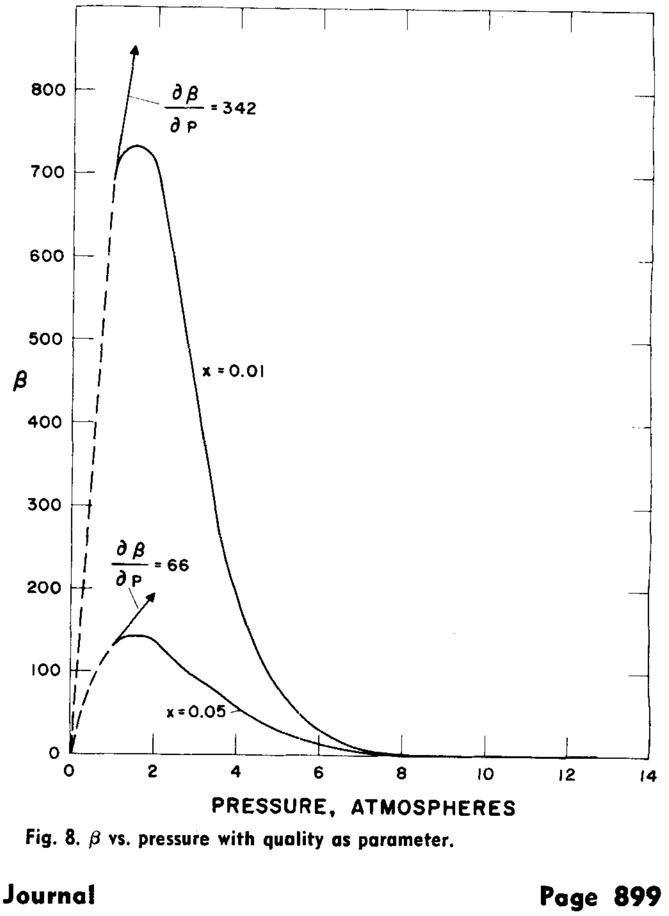

Fig. 8. β vs. pressure with quality as parameter.

In consideration of the above discussion the Lockhart and Martinelli holdup correlation was used, due to the obvious convenience of starting with the Lockhart and Martinelli $R_l - X$, the consistency in dealing solely with a single model, and because the Hughmark correlation must be tied to a specific geometry through the Reynolds and Froude numbers. The Lockhart and Martinelli values of R_l are used for computation of α and β values at 1 atm.; these, together with the well defined values of α at the critical pressure, serve to determine all the values of α for the entire $\alpha - P - x$ map and the value of $\partial\Phi/\partial P$ at 1 atm.

The β map as shown in Figures 8, 9, and 10 shows maxima at pressures approximately equal to 1.5 atm. for all curves independent of quality. Apparent minima exist in the vicinity of 9 atm. for the higher quality curves. Equation (11) can be set equal to zero and manipulated to give the trivial result of $\beta = 1$ at the minimum. Other explicit expressions for maxima and minima are not determinable from Equation (11) as Φ, α, and χ are not explicit functions of pressure. Because of the uncertainties in α, etc., it is not clear that the maxima in the curves are all at the same pressure. The apparent minima near 9 atm. are even less well defined since the calculations as outlined above were not executed in detail above 7 atm. This is discussed below.

Figure 10 shows the inversion of β which, for low pressures, increases with decreasing quality until a very low quality is reached when a sharp decline in β occurs with a decrease to $\beta = 0$ at $x = 0$. β was estimated at 1 atm. with Equation (3) and the $R_l - X$ correlation of Lockhart and Martinelli. Note: $R_g = 1 - R_l$. β reaches a maximum of approximately 5,500 for $x = 0.0015$ and is down to about 500 at $x = 0.0010$.

The broken lines for the $\beta - P$ curves below $P = 1$ have been added to indicate that as the pressure is reduced the system must tend toward the gaseous state and β becomes one for all qualities. (In reality, it must be recognized that if liquid is present at quite low pressures, the equilibrium state will give rise to solid formation.) Under the same conditions α becomes zero.

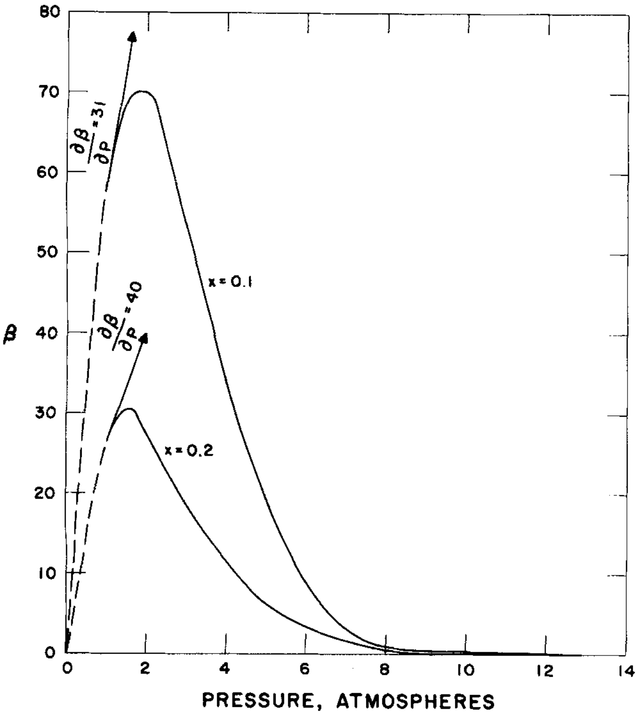

Fig. 9. β vs. pressure with quality as parameter.

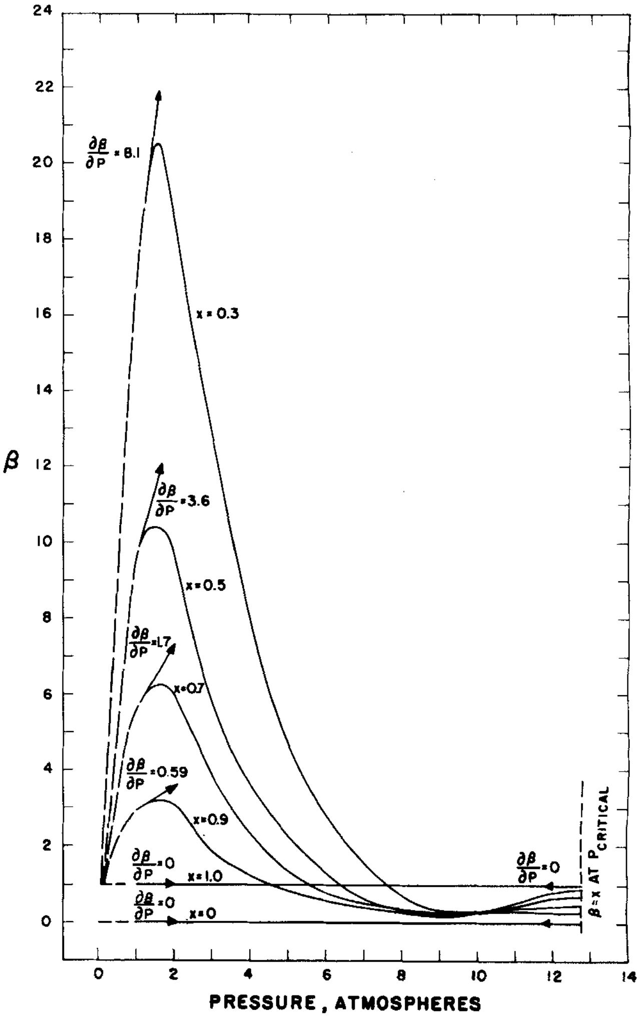

Fig. 10. β vs. pressure with quality as parameter.

α and β take on additional physical significance in view of the information now developed. They become indicators of surface to volume ratio. Consider the defining equation for β:

$$\beta = \frac{A_g}{\frac{\pi}{4} D_g^2} = \frac{R_g D_p^2}{D_g^2} \tag{13}$$

or

$$\beta D_g^2 = R_g D_p^2 \tag{14}$$

From Figure 12, the observation is made that for a given quality, R_g at the most ($x \approx 0.01$) varies roughly 44 fold with pressure while β varies 700 fold as observed from Figure 8. In the other extreme ($x \approx 0.5$) R_g varies roughly a factor of five while β varies 80 fold. For either case the right hand side of Equation (14) is, by comparison, a slowly varying value for a given D_p. β then becomes a probable indicator of the number of effective gas volumes of a circular cross section with diameter D_g. Obviously, the vapor volumes do not have the idealized single cylindrical shape but must have cross sectional areas which, on the average, are equivalent to a number of cylinders of diameter D_g. β, then, is an indicator of the state of subdivision of the flowing vapor and hence is also a measure of vapor-liquid (and vapor-pipe) interfacial surface to vapor volume. This physical interpretation of β makes the analysis of the $\beta - P$ curves more direct and removes, to some extent, from the Martinelli model the stigma of being limited to a single cylindrical and/or annular flow model.

This interpretation of β should be qualified with the recognition that those uncertainties of the two-phase

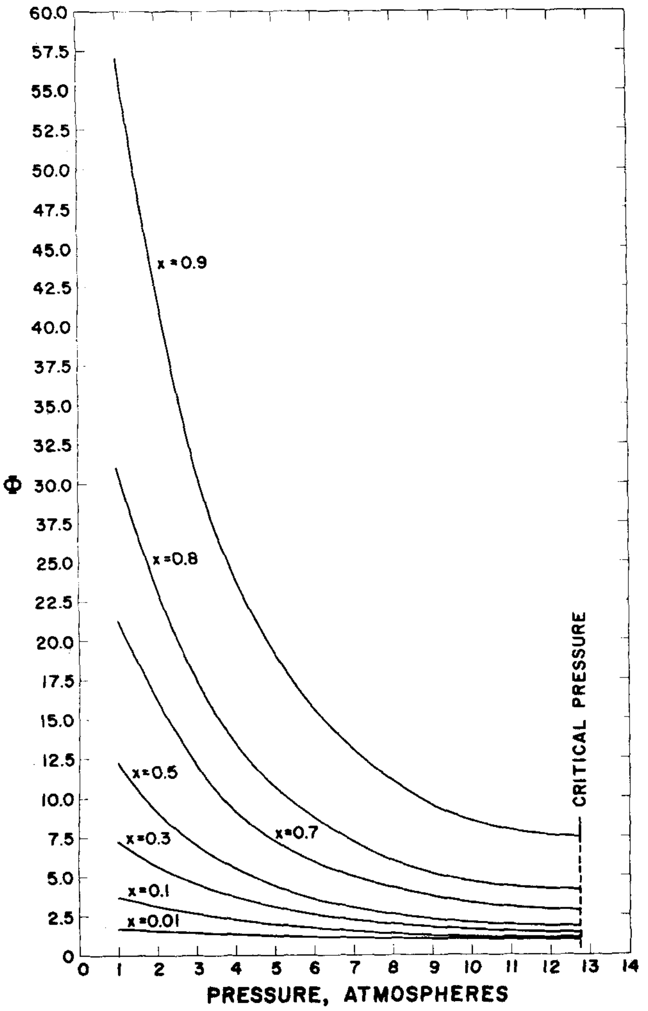

Fig. 11. Φ vs. pressure with quality as parameter.

model, such as the effect of the relative velocity of the phases, and flow patterns have been absorbed in β. Since α and β are the parameters of the model, they must necessarily contain all information not otherwise explicitly included. The Martinelli model does not, for instance, explicitly consider the effect of flow patterns, and no experimental information on flow patterns was available for the data reported below. The selection of α, Figure 6, forces the division of such information between α and β in an unknown way. The subsequent and previous discussion of α and β should also be considered in this light.

Figures 8, 9, and 10 show, for constant quality, an increase in surface to volume ratio of vapor in going from low pressures to about 1.5 atm. This indicates an increase in subdivision of the vapor phase. Above 1.5 atm. coalescence occurs with a rapid decrease in surface to volume ratio. The absolute level of surface to volume ratio is dependent upon quality and is generally less with increased quality. This is possibly a consequence of increased contact with coalescence between portions of gas flowing or a rearrangement of a continuous gas phase to minimize surface energies between the two phases.

The minima in the high quality $\beta - P$ curves near 9 atm. are primarily a consequence of expanding the single phase boundary condition at the critical point into a line. For a consistent solution of Equations (4) through (7) at the critical point, β was shown to become identical to the quality, Equation (9). This is the result of maintaining the concept of quality along the coexistence curve and at the critical point. Values for β at pressures somewhat below critical tend toward a low value and then diverge to satisfy the critical point conditions, $\beta = x$ and $\partial\beta/\partial P = 0$.

The defining equation for α is

$$\alpha = \frac{R_l D_p^2}{D_l^2} \tag{15}$$

α is a measure of the surface to volume ratio for the liquid phase. The α map of Figure 6 indicates, in contrast to the vapor phase β, a low state of subdivision and, in general, a smaller surface to volume ratio for all qualities. Contrary to the vapor, the liquid surface to volume ratio increases, but only slightly, with increasing quality. Since α does not vary over large extremes, this trend is determined by computing the index α/R_l in detail. Also contrary to the vapor modulus β, the liquid modulus shows a consistent increase in α with pressure to the critical point value; however, the surface to volume ratio for the liquid decreases somewhat with increasing pressure over the entire pressure range.

The $\Phi - X - P - x$ surface of Figure 12 is considerably different in character than that postulated by Martinelli and Nelson (6) for water. It contains certain features which are reasonable in terms of the properties of all fluids. These features support the validity of the approach presented in this paper and indirectly give substance to the assumed α map.

As a flowing two-phase system approaches the critical point condition, the pressure drop contributions of each phase become more nearly equal as do the densities and viscosities. The two-phase pressure drop should disappear at the critical condition with $\partial\Phi/\partial P = 0$ as per Figure 11. This characteristic is met if $\partial\Phi/\partial X$ is also zero (for constant quality) at the critical point. This feature is exhibited in Figure 12. As a consequence of $\partial\Phi/\partial X = 0$, which can be shown analytically, some isobars of $\Phi - X$ will lie to the left (see the 11 atm. $\Phi - X$ curve) of the critical condition $\Phi - X$ curve.

The uncertainties in the densities and viscosities and their effect upon the calculations were examined. The densities are known to very high precision and can be forthwith dismissed as a source of error. The liquid viscosities in the intermediate pressure region were obtained by an interpolation technique (5) and might be in error as much as 20 to 30% near the critical condition. This in turn would result in an error of only a few percent in the calculations. Nevertheless, because of the viscosity uncertainty, which increases as pressure is increased, the iterative computation of the $\Phi - P$ curves was limited to pressures equal to or below 7 atm., and the curves were interpolated from 7 atm. to the well determined critical point condition. Figure 11 shows these curves and it is apparent that a good interpolation over this range is readily accomplished. The β values as computed from Equation (10) are sensitive to the interpolated Φ values, hence

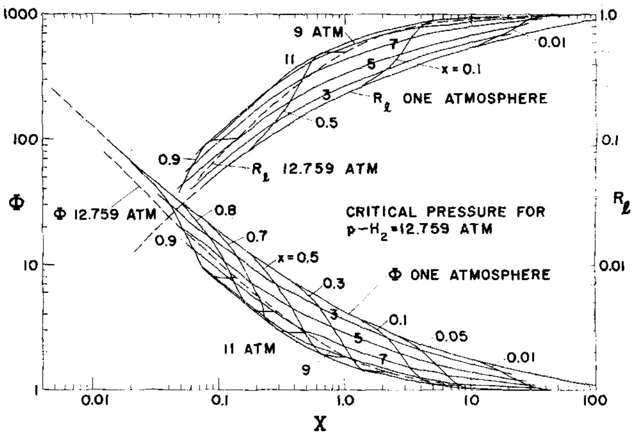

Fig. 12. $\Phi - X - P - x$ surface and $R_l - X - P - x$ surface.

the indefinite nature of the minima near 9 atm. of Figure 10.

COMPARISON WITH EXPERIMENTAL DATA

The liquid hydrogen data of Richards, et al. (*10*), originally taken for purposes other than comparison with the Martinelli model, show a wide scatter. Values for Φ at pressures of 15.1, 17.4, and 32.2 lb./sq.in.abs. lie about the same distance below the 1 atm. Martinelli curve and values at 15.1, 37, and 39 lb./sq.in.abs. lie considerably above the curve (*11*). No systematic trend of the data with pressure exists. A diameter affect does exist. Large Φ values are generally associated with large diameters. These data are not useful for comparison.

Experiments with a brass tube, 3/16 in. I.D. and a mass flow of 0.038 lb./sec. were examined. These data represent two-phase flow with heat transfer ranging from all film boiling to nucleate boiling as the subcooling length becomes longer. The observed pressure drop was increased by 2 lb./sq.in. to account for a zero drift in one of the pressure transducer amplifiers. The calculated and observed pressure drops are given in Table 3. The liquid entered the test section at zero quality and the calculation accounted for subcooling. Exit qualities were computed from a heat balance and measured wall temperatures. The computation was also performed with a zero friction term and a momentum contribution was calculated. The tabular observed and calculated two-phase pressure drops include the momentum effect. Because of the interaction between friction and momentum effects, the momentum term listed in the table cannot be subtracted directly from the overall pressure drop to obtain the two-phase friction contribution for quality greater than 0.50. The computation was carried out on an incremental basis over the length of the test section by numerical computer solution with an assumed axial symmetry. The calculated exit qualities and momentum effects are also listed in Table 3 along with the percent difference between calculated and observed pressure drop based upon the observed value. The agreement between observed and calculated values is good.

TABLE 3. TWO-PHASE PRESSURE DROP DATA FOR PARA-HYDROGEN

Inlet Pressure	$\Delta P_{\text{TP, observed}}$	$\Delta P_{\text{TP, calc.}}$	$\Delta P_{\text{momentum calc.}}$	$x_{\text{exit, calc.}}$	Subcooling Length	% Difference
184 lb./sq.in.abs.	28.5 lb./sq.in.	20.5 lb./sq.in.	5.1 lb./sq.in.	0.81	4 in.	—28
154	24.0	21.0	4.5	0.40	6	—12.5
111	18.0	17.6	2.4	0.22	16	—2.3
91	16.0	19.1	2.2	0.13	28	19.4
80	16.0	18.2	1.7	0.08	30	13.7
72	15.5	16.2	0.9	0.04	34	4.5
69	15.0	15.5	0.2	0.02	42	3.3
68	15.0	15.0	0.4	0.01	46	0
67	15.0	14.5	0.1	zero	—	—3.3

ACKNOWLEDGMENT

The author would like to express his appreciation to John Bartlit and Kenneth Williamson for constructive discussions; to Geoffrey Watts, who supplied the experimental data; to Rodney Thurston, who made the machine computations of the pressure drop using the experimental data; and to Kieth Zeigler for his assistance with the least square computation of Equation (12*a*).

This work was performed under the auspices of the U.S. Atomic Energy Commission.

NOTATION

A = cross sectional area of either phase
B = $\Phi^{0.833} - \alpha^{0.25}$, Equation (10)
D = hydraulic diameter of either phase or I.D. of pipe
L = length of pipe
P = pressure, atm.
R = volume fraction of either phase
x = weight fraction gas, quality
Φ = Martinelli two-phase friction factor
X = $\left[\left(\frac{\rho_g}{\rho_l}\right)^{0.555}\left(\frac{\mu_l}{\mu_g}\right)^{0.111}\left(\frac{1-x}{x}\right)\right]^{0.9}$ = Martinelli correlating parameter

Greek Letters

α = ratio of cross sectional area of liquid to an idealized cylindrical area
β = ratio of cross sectional area of vapor to an idealized cylindrical area
μ = viscosity of either phase
ρ = density of either phase
χ = $\left(\frac{\rho_g}{\rho_l}\right)^{0.555}\left(\frac{\mu_l}{\mu_g}\right)^{0.111}\left(\frac{1-x}{x}\right)$ = Martinelli correlating parameter

Subscripts

SP = single phase
TP = two phase
g = gas or vapor phase
l = liquid phase
p = pipe
ltt = liquid phase flowing alone, both phases turbulent

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Manuscript received July 31, 1967; revision received December 1, 1967; paper accepted December 4, 1967.